Majorana Ferimons

Horan Tsui

1 Majorana Fermions

We adopt the notation in accordance with Peskin, that is, the Dirac representation, for example, $\sigma^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}$ and $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}$.

1.1 Another Solution

Although all confirmed fermions of SM so far are Dirac fermions, but some structures would be more clear from the viewpoint proposed by Weyl:

$$i\sigma^\mu \partial_\mu \psi_L = m\psi_R \tag{1}$$

$$i\bar{\sigma}^\mu \partial_\mu \psi_R = m\psi_L \tag{2}$$

ψ_L and ψ_R satisfy E.o.M (1) and (2) respectively. But not only do they satisfy such E.O.M.s: Conjugate (1), we have:

$$\begin{aligned} -i\sigma^{*\mu}\partial_\mu \psi_L^* &= m\psi_R^* \\ -i\sigma^2\sigma^{*\mu}\sigma^2\partial_\mu(\sigma^2\psi_L^*) &= m\sigma^2\psi_R^* \\ -i\bar{\sigma}^\mu\partial_\mu(\sigma^2\psi_L^*) &= m\sigma^2\psi_R^* \\ i\bar{\sigma}^\mu\partial_\mu(i\sigma^2\psi_L^*) &= m(-i\sigma^2\psi_R^*) \end{aligned} \tag{3}$$

where we used the identity $\sigma^2\sigma^{*\mu}\sigma^2 = \bar{\sigma}^\mu$, and $\sigma^2\sigma^2 = I$.

(3) means a pair of new objects, $i\sigma^2\psi_L^*$ and $-i\sigma^2\psi_R^*$, constructed from (1) while satisfying (2). These stuff are also the solutions of Weyl equations. In contrast, scalar bosons' E.o.M. is relatively simple, without such non-trivial alternatives.

From the figuration of (3), it's obviously that $i\sigma^2\psi_L^*$ is right-handed, and $-i\sigma^2\psi_R^*$ is left-handed. We would denote them as χ_R and χ_L respectively.

Another pair of solution of Weyl equations is $\begin{pmatrix} -i\sigma^2\psi_R^* \\ i\sigma^2\psi_L^* \end{pmatrix} = -i\gamma^2\psi^* = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \chi$

1.2 Charge Conjugation

What's the relationship between χ and ψ? Or equivalently, what's the $-i\gamma^2\psi^*$? The answer is hidden in the field representation:

$$\psi = \psi_L + \psi_R = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_s (a_p u_p^s e^{ik\cdot r} + b_p^\dagger v_p^s e^{-ik\cdot r}) \tag{4}$$

hence:

$$\begin{aligned} -i\gamma^2\psi^* &\sim \sum_s (-i\gamma^2)(a_p^\dagger u_p^{*s} e^{-ik\cdot r} + b_p v_p^{*s} e^{ik\cdot r}) \\ &= \sum_s (a_p^\dagger(-i\gamma^2 u_p^{*s})e^{-ik\cdot r} + b_p(-i\gamma^2 v_p^{*s})e^{ik\cdot r}) \\ &= \sum_s (a_p^\dagger v_p^s e^{-ik\cdot r} + b_p u_p^s e^{ik\cdot r}) \end{aligned} \tag{5}$$

where we used the definition that $u_p^s = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}$, $v_p^s = \begin{pmatrix} \sqrt{p\cdot\sigma}(-i\sigma^2\xi^{*s}) \\ \sqrt{-p\cdot\bar{\sigma}}(-i\sigma^2\xi^{*s}) \end{pmatrix}$, and, consequently, the identity that $-i\gamma^2 u_p^{*s} = v_p^s$, and $-i\gamma^s v_p^{*s} = u_p^s$. Compared with (4), we find now $a^\dagger$ will generate an anti-particle which would be generated by $b^\dagger$ before. Hence the exchange between anti-particles and particles.

This process is called charge conjugation, which is formally denoted as $C\psi C^\dagger$, under the Dirac representation, it amounts to $C\bar{\psi}^T$, where $C = -i\gamma^2\gamma^0$.

χ_R is the conjugated counterpart of ψ_L, and vice versa, so does another handed part.

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \xrightarrow{C.C.} \begin{pmatrix} -i\sigma^2\psi_R^* \\ i\sigma^2\psi_L^* \end{pmatrix} = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

1.3 Ettore Majorana

So far, nothing beyond that of Dirac or Weyl, but here comes the great Majorana, who combines together the χ part and the ψ part. (Someone like to rewrite χ as ψ^c to illustrate their relationship, we will adopt this notation henceforth.)

A new somewhat fancy combination proposed by Majorana:

$$\Psi_1 = \begin{pmatrix} \psi_L \\ \psi_L^c \end{pmatrix} = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix}, \Psi_2 = \begin{pmatrix} \psi_R^c \\ \psi_R \end{pmatrix} = \begin{pmatrix} -i\psi_R^* \\ \psi_R \end{pmatrix} \tag{6}$$

It's not hard to prove that $C\Psi_1 C^\dagger = \Psi_1$, so does Ψ_2. This reveals the meaning of such combination - a kind of new particle whose anti-partner is itself! Those particles of the form of Ψ_1 or Ψ_2 are called the majorana fermions, in the memory of Majorana, an ineffable Italian physicist with a mysterious and young death.

Some remarks:

1. A majorana fermion must be charge-neutral since charge conjugation will flip a particle's charge (recall the complex conjugation part).

2. Although a majorana fermion should be neutral as a 4-component spinor, the left-handed and right-handed part could be oppositely charged. This is unique.

   A Dirac fermion's left-handed part has no such relationship with the right-handed part, for instance, the left-handed part of electron, e_L, has -1 electronic charge and $-\frac{1}{2}$ isotopic charge, while e_R has -1 electronic charge but 0 isotopic charge. It's the consequence of parity symmetry breaking in weak interaction.

   However, if we set the left-handed part ψ_L of a majorana fermion Ψ_1 as of 0 electric charge and $+\frac{1}{2}$ isotopic charge, then ψ_L^c could only have 0 electric charge and $-\frac{1}{2}$ isotopic charge.

   A right-handed spinor is essentially different from a left-handed spinor's conjugation.

1.4 A Heuristic

Let's represent Ψ and ψ in a vivid way.

First, a general fermion always consists of a left-handed part and a right-handed part:

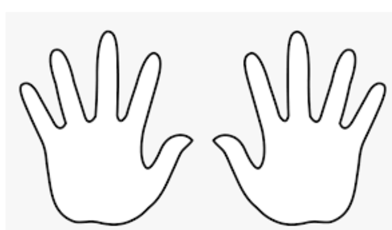

Figure 1: A general fermion

A Dirac Fermion has ψ_L and ψ_R, we represent them as palms of a left hand and a right hand respectively, its charge conjugation ψ^c would be backs of two hands.

Figure 2: A Dirac fermion and its anti version

A Majorana fermion of the form Ψ_2 would be:

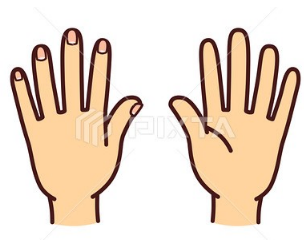

Figure 3: Majorana fermion