Hawking Radiation for Massive Scalar Particles

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Brief history

TABLE Timeline

Einstein 1915	The bending of light. Particle propagation.
Wheeler, Brill, et al. 1957	Stability of Schwarzschild BHs. Wave propagation.
Hildreth 1964	Massless scalar particles' cross section.

Table Timeline

Teukolsky, Press, Brill 1972-1974	The separation of variables of scalar waves in Kerr by NP form.
Chandrasekhar, Chrzanowski, Futterman et al. 1975-1981	The separation for massless spin-1/2, spin-1 and spin-2 waves.

Curved spacetime QFT

The particle world is connected with the field world by the mode expansion:

$$\Phi(x) = \sum_i (af_i + bf_i^*) \tag{1}$$

f and f^* should satisfy the E.o.M. and some constraints, like the K-K norm and positive/negative frequency requirement.

These requirements are clear and definite in Minkovsky spacetime, but are generally not in curved spacetime.

No translation and boost symmetry makes it impossible to have an translation-invariant decomposition, a pure positive frequency mode will not be a positive one but a mixture after translating.

Thus the particle world blurs. The Hawking radiation rises from this ruin.

Nonetheless, the field world still works unless the curvature is small compared to Planck scale:

The energy density $[E/V] = [\rho] = L^{-4}$, and the Ricci curvature $[R] = L^{-2}$, therefore $\Delta\rho \sim R^2$.

$\Delta\rho$ in trun affects the metric and the R. This loop goes forever.

The curved spacetime QFT works well when R is small. In this region, we could use the well-behaved fields to discuss without considering the second quantization.

Teukolsky equation

Scalars in curved spacetime satisfy:

$$\nabla_\mu \nabla^\mu \Phi = \mu^2 \Phi \tag{2}$$

In spherically symmetrical spacetime, it's obvious that we can separate the variables of wave functions with certain ω as:

$$\Phi(x) = \sum_{l,m} e^{-i\omega t} Y_l^m(\theta, M) e^{im\phi} R_{\omega l}(r, M) \tag{3}$$

where M is the BH's mass.

In 1973, Teukolsky has proved a remarkable result that Φ can also be expressed as (3) with a slightly modified angular part.

$$\Phi(x) = \sum_{l,m} e^{-i\omega t} S_l^m(\theta, \phi M) R_{\omega l}(r, M) \tag{4}$$

where S_l^m satisfies:

$$[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta}) + K_{lm} + M^2(\mu^2 - \omega^2)sin^2\theta - \frac{m^2}{sin\theta^2}]S_l^m = 0 \tag{5}$$

where K_{lm} represents the eigenvalue.

In general, all radial equations can be rewrited as:

$$\frac{d^2}{d(r^*)^2}R + (\omega^2 - V)R = 0 \tag{6}$$

where r^* is the tortoise coordinate.
To illustrate, let's consider the Schwarzschild spacetime, and rewrite $R_{\omega l}$ as $\frac{\psi_{\omega l}}{r}$:

$$\nabla_\mu \nabla^\mu \Phi = \frac{-\omega^2}{f}\Phi - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f \frac{\partial}{\partial r})\phi - \frac{1}{r^2}\frac{1}{sin^2\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta})\Phi - \frac{1}{sin^2\theta r^2}\frac{\partial^2}{\partial\phi^2}\Phi = \mu^2\Phi \tag{7}$$

Effective potential

After separation:

$$f\frac{d}{dr}(f\frac{d}{dr})\psi+\omega^2\psi-f\frac{l(l+1)}{r^2}-f\mu^2-f\frac{2M}{r^3}=\frac{d^2}{d(r^*)^2}\psi+(\omega^2-V)\psi=0 \tag{8}$$

where $V=f(\frac{l(l+1)}{r^2}+\mu^2+\frac{2M}{r^3})$, the last term comes from rewrite r as r^*.

The angular part is regular.

For massless scalars, the maximum of V is:

$$r = 3M \frac{l(l+1) - 1 + \sqrt{[l(l+1)]^2 + 32/9 l(l+1)}}{2l(l+1)} \tag{9}$$

which approaches to 3M for large l asymptotically. Slightly larger than the classical limit 2.6M (BH shadow area $27\pi M^2$).

It goes to 0 near the horizon and infinity.

Asymptotic form

Equation (8) is solvable near the horizon($V \to 0$) and infinity($r* \to r, V \to \mu^2$):

$$\psi_{\omega l} = \begin{cases} A_{\omega l} e^{-i\omega r^*}, & r \to r+, \\ e^{-i\omega v r} + R_{\omega l} e^{-\omega v r}, & r \to \infty. \end{cases} \tag{10}$$

where $v = \sqrt{1 - \frac{\mu^2}{\omega^2}}$.And we have used the boundary condition that there is only ingoing wave at the horizon.

(6) has no first order derivative, therefore the Wronskian is a constant:

$$\dot{W}(\psi, \psi*) = (\dot{\psi}\dot{\psi}* + \psi\ddot{\psi}^{*}) - (\dot{\psi}\dot{\psi}* + \psi^{*}\ddot{\psi}) = 0 \tag{11}$$

therefore, $W_{r+} = W_{\infty}$:

$$|A_{\omega l}|^2(2i\omega) = 2i\omega v - 2i\omega v|R_{\omega l}|^2$$

this amounts to

$$\frac{|A_{\omega l}|^2}{v} + |R_{\omega l}|^2 = 1 \tag{12}$$

Actually, $\frac{|A_{\omega l}|}{\sqrt{v}}$ is $|T_{\omega l}|$, the transition rate.

All calculations of BH emission rates come down to calculating $|T_{\omega l}|^2$ or $|R_{\omega l}|^2$, which are the methods of Unruh and Page respectively.

More specifically, the cross-section $\sigma = \frac{\pi}{k^2}\sum_l (2l+1)|T_{\omega l}|^2 = \frac{\pi}{k^2}\sum_l (2l+1)(1-|S_{\omega l}|^2)$.

Review

	Page	Unruh
Object	Massless spin-0, 1/2 Schwarzschild, Kerr, RN	Massive spin-0, 1/2 Schwarzschild
Method	when $\omega < 2M$: Analytically calculate $\lvert R_{\omega l}\rvert^2$ at infinity when $\omega > 2M$: numerically	when $\omega < 2M$: Analytically calculate $\lvert T_{\omega l}\rvert^2$ by the flux ratio, infinity to horizon
Result	$\sigma \sim A$ when $\omega \to 0$	$\sigma \sim \frac{A}{v}$ when $\omega \to \mu$

Hawking radiation

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + Qd\Phi \tag{13}$$

"It should not be thought unreasonable that a black hole, which is an excited state of the gravitational field, should decay quantum mechanically and that, because of quantum fluctuation of the metric, energy should be able to tunnel out of the potential well of a black hole." - Hawking 1975

Gray body factor

In Minkovsky spacetime, thermo-rate:

$$2\pi \frac{dN}{dtd\omega} = \frac{1}{e^{(\omega+...)/T} \pm 1}$$

In curved spacetime, however, particles are subject to gravitational attraction or refection, an effect from curvature:

$$2\pi \frac{dN}{dtd\omega} = \langle N_{\omega mql} \rangle = \frac{\Gamma_{\omega mql}}{e^{2\pi\kappa^{-1}(\omega - m\Omega - q\Phi)} \pm 1} \tag{14}$$

From the QFT viewpoint, it resembles to the forward scattering and backward scattering, the total effect encodes in the cross section:

$$\frac{dN}{dt} = \int \sum_{m,q,l} \frac{\Gamma_{\omega mql}}{e^{2\pi\kappa^{-1}(\omega - m\Omega - q\Phi)} \pm 1} d\omega \tag{15}$$

q: electronic charge;
m: angular momentum number;
l: spheroidal harmonics;
μ: particle mass.

Cross section

$\Gamma_{\omega mql}$ is exactly the absorption probability of a certain mode, thus:

$$\sigma = \frac{\pi}{k^2}\sum_l (2l+1)|T_l|^2 = \frac{\pi}{k^2}\sum_{m,q,l}\Gamma_{\omega mql} \tag{16}$$

Therefore, (2) becomes:

$$2\pi\frac{dN}{dt} = \int \frac{\sigma k^2/\pi}{e^{2\pi\kappa^{-1}(\omega - m\Omega - q\Phi)} \pm 1} d\omega \tag{17}$$

Bound state
Superradiant (in)stability
Scalar hair

ω and Ω

Thanks!