Exotic Transition

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Outline

Introduction

- What's the **meaning of 'exotic'** ?

- **Why** could non-commutativity induce exotic transitions?

Short Review

Model: $\mathcal{B}_{\chi\hat{n}}$:

$$f(x) \star g(x) = e^{-\frac{i}{2}\chi P_0 \wedge J_{\hat{n}}} f(x+\eta) g(x+\eta)|_{\eta\to 0}$$
$$= e^{-\frac{i}{2}\chi (P_0 \otimes J_{\hat{n}} - J_{\hat{n}} \otimes P_0)} f(x+\eta) g(x+\eta)|_{\eta\to 0}$$

One needs to replace the normal product $\cdot$ with the $\star$ product.

For energy and angular-momentum eigenstates:

$$\begin{aligned}
&\Delta_{\chi\hat{n}}(\sigma)|1,2\rangle \\
=&F(\sigma\otimes\sigma)F^{-1}|1\rangle\otimes|2\rangle \\
=&|\sigma(1)\rangle\otimes e^{-\frac{i}{2}\chi(E_{\sigma(1)}J_{\hat{n}}-J_{\sigma(1)}P_0)}\sigma e^{\frac{i}{2}\chi(E_1 J_{\hat{n}}-J_1 P_0)}|2\rangle \\
=&e^{-\frac{i}{2}\chi(E_{\sigma(1)}J_{\sigma(2)}-J_{\sigma(1)}E_{\sigma(2)})}e^{\frac{i}{2}\chi(E_1 J_2 - J_1 E_2)}|\sigma(1),\sigma(2)\rangle \\
=&e^{-i\chi(E_2 J_1 - E_1 J_2)}|2,1\rangle
\end{aligned}$$

Adopt a notion:

$$F \to F(x,y) := e^{-\frac{i}{2}\chi(E_x J_y - J_x E_y)}$$

$$F(x,y) = F^*(y,x) = F^{-1}(y,x)$$

Then

$$\Delta_{\chi\hat{n}}(\sigma)|1,2\rangle = F^2(2,1)|2,1\rangle$$

Or

$$\Delta_{\chi\hat{n}}(\sigma)F(1,2)|1,2\rangle = F(2,1)|2,1\rangle$$

Intrinsic phase factors in front of states.

Fock space

We have proved:

$$|\psi\rangle = \frac{1}{\sqrt{N!}} \sum_{\{n\}} sgn_{\{n\}} (I \otimes I \otimes I ... \Delta_{\chi \hat{n}}) ... \Delta_{\chi \hat{n}} (\sigma_{\{n\}}) |1, 2, 3...N\rangle$$

$$= \frac{1}{\sqrt{N!}} \sum_{\{n\}} sgn_{\{n\}} \amalg_{i<j,1}^{N} F[\sigma_{\{n\}}(i), \sigma_{\{n\}}(j)] \times$$

$$|\sigma_{\{n\}}(1), \sigma_{\{n\}}(2), ...\sigma_{\{n\}}(N)\rangle$$

Proof

Mathematical Induction: (n, appearing times of $\Delta_{\chi\hat{n}}$)
A trick:

$$
\begin{aligned}
&F(\sigma_a \otimes \sigma_b)F^{-1} \\
=&\sigma_a \otimes F[\sigma_a, ?]\sigma_b F^{-1}(_a, ?) \\
=&\sigma_a \otimes \sigma_b^{'}
\end{aligned}
$$

Red is an operator.
The first slot of $F(?, ?)$ is determined by the front σ, and leave the second slot to the latter, making it a new $\sigma^{'}$ carrying a phase-producing operator.

$$\Delta_{\chi\hat{n}}(\sigma)|1,2\rangle = \sigma_a \otimes \sigma_b^{'}|1,2\rangle$$

let a act on $|1\rangle$ and b act on $|2\rangle$

$$\begin{aligned} &= |\sigma_a(1)\rangle F[\sigma_a(1), \sigma_b(2)] F^{-1}(1,2) |\sigma_b(2)\rangle \\ &= F[\sigma(1), \sigma(2)] F^{-1}(1,2) |2,1\rangle \end{aligned}$$

$$(I \otimes \Delta_{\chi\hat{n}})\Delta_{\chi\hat{n}}(\sigma) = (I \otimes \Delta_{\chi\hat{n}})(\sigma_a \otimes \sigma_b^{'}) = \sigma_a \otimes (F\sigma_{b1}^{'} \otimes \sigma_{b2}^{'}F^{-1})$$

$$F\sigma_{b1}^{'} \otimes \sigma_{b2}^{'}F^{-1} = \sigma_{b1}^{'} \otimes F[\sigma_{b1}, ?]\sigma_{b2}^{'}F^{-1}(_{b1}, ?) = \sigma_{b1}^{'} \otimes \sigma_{b2}^{''}$$

Let a act on $|1\rangle$, b1 act on $|2\rangle$, and b2 act on $|3\rangle$, we have

$F[\sigma_a(1), \sigma_{b1}(2)]F^{-1}(1,2)$ from $\sigma_{b1}^{'}$

$F[\sigma_a(1), \sigma_{b2}(3)]F[\sigma_{b1}(2), \sigma_{b2}(3)]F^{-1}(1,3)F^{-1}(2,3)$ from $\sigma_{b2}^{''}$

and

$$(I \otimes \Delta_{\chi\hat{n}})\Delta_{\chi\hat{n}}(\sigma) = \sigma \otimes \sigma' \otimes \sigma^{''}$$

1 from σ_a

$F[\sigma_a(1), \sigma_{b1}(2)]$... from σ'_{b1}

$F[\sigma_a(1), \sigma_{b2,1}(3)]F[\sigma_{b1}(2), \sigma_{b2,1}(3)]$... from $\sigma''_{b2,1}$

$F[\sigma_a(1), \sigma_{b2,2}(4)]F[\sigma_{b1}(2), \sigma_{b2,2}(4)]F[\sigma_{b2,1}(3), \sigma_{b2,2}(4)]$... from $\sigma'''_{b2,2}$

...

Subscripts of σ should not be confusing since they are only used to denote the inheritance relationship and can be omitted once acting on certain states.

For $\sigma \otimes \sigma^{'} \otimes \sigma^{''} \otimes \sigma^{'''} \otimes ...$

$F[\sigma(1), \sigma(2)]... \times$

$F[\sigma(1), \sigma(3)]F[\sigma(2), \sigma(3)]... \times$

$F[\sigma(1), \sigma(4)]F[\sigma(2), \sigma(4)]F[\sigma(3), \sigma(4)]... \times$

...

n=1: $\Delta_{\chi\hat{n}}(\sigma) = \sigma \otimes \sigma^{'}$, ✓

n=N-1: suppose ✓

n=N:

$$\underbrace{(I \otimes I \otimes I ... \Delta_{\chi\hat{n}})(I \otimes I ... \Delta_{\chi\hat{n}}) ... \Delta_{\chi\hat{n}}}_{\text{N}}(\sigma)$$

$$=(I \otimes I \otimes I ... \Delta_{\chi\hat{n}})(\sigma \otimes \sigma' \otimes \sigma^{''} ... \sigma^{'''\cdots})$$

$$=\sigma \otimes \sigma' \otimes \sigma^{''} ... F(\sigma^{'''\cdots} \otimes \sigma^{'''\cdots})F^{-1}$$

$$=\sigma \otimes \sigma' \otimes \sigma^{''} ... \sigma^{'''\cdots} \otimes \sigma^{''''\cdots}$$

Q.E.D.

The initial 3-Particle state

X: $\uparrow$, ...

...

1S: $\uparrow$, $\downarrow$

$$
\begin{aligned}
|\psi\rangle = \frac{1}{\sqrt{3!}}\{ & e^{-\frac{i}{2}\chi E_{1s}(2J_X-1)}|1+,1-,X+\rangle \\
& -e^{-\frac{i}{2}\chi E_{1s}(-2J_X+1)}|X+,1-,1+\rangle \\
& +e^{-\frac{i}{2}\chi E_{1s}(-1-2J_X)}|X+,1+,1-\rangle \\
& -e^{-\frac{i}{2}\chi(-E_{1s}-E_X)}|1+,X+,1-\rangle \\
& +e^{-\frac{i}{2}\chi(E_{1s}+E_X)}|1-,X+,1+\rangle \\
& -e^{-\frac{i}{2}\chi E_{1s}(1+2J_X)}|1-,1+,X+\rangle\}
\end{aligned}
$$

The final 3-particle state

1S: $\downarrow$, $\uparrow$, $\downarrow$

$$|\phi\rangle_{\textit{fermion}} \equiv 0$$

A general statement:
In non-commutative spacetime with the Drinfel'd twist type models, like θ-Poincare and $\mathcal{B}_{\chi\hat{n}}$, identical fermions (energy, angular momentum, *etc*) are still forbidden.

A corrected PEP: The identical is not only for states, but also for the front phase factors.
The identical fermions are forbidden.

Boson-Fermion Transition

In the regular spacetime and SM, fermions can indeed become bosons, but they should be different particles, like $e^+e^- \to \gamma$.

The fermionic/bosonic-ness of particles is definite. No intersection. **No potential can turn a fermion into a boson** with the same angular momentum (including spin), energy, and charge.

SUSY partners have a different spin.

In non-commutative spacetime, however, we can.

Suppose $a_p^\dagger$ is the creation operator for a kind of fermion, and $b_p^\dagger$ for the same-type boson.

In regular spacetime:
If there is a transition turning a fermion into a same-type boson,

$$\langle 0|b_m b_n V a_p^\dagger a_q^\dagger|0\rangle \neq 0$$

Same-type: particles are correspondent one-to-one. For fermions, $\langle p,q|p,q\rangle = \langle q,p|q,p\rangle = -\langle q,p|p,q\rangle = 1$
Then it should be:

$$\langle 0|b_m b_n V a_p^\dagger a_q^\dagger|0\rangle = \langle 0|b_n b_m V a_q^\dagger a_p^\dagger|0\rangle$$

However,

$$b_n b_m = b_m b_n$$
$$a_p^\dagger a_q^\dagger = -a_q^\dagger a_p^\dagger$$

then

$$\langle 0| b_m b_n V a_p^\dagger a_q^\dagger |0\rangle = -\langle 0| b_n b_m V a_q^\dagger a_p^\dagger |0\rangle$$

Such transition is prohibited by the spin-statistics.

In non-commutative spacetime, $\cdot \to \star$,

$$\langle 0| b_m \star b_n \star V \star a_p^\dagger \star a_q^\dagger |0\rangle$$

V could be a complex combination of creation/annihilation operators, but to illustrate, we sloppily treat it as one.
By the twisted permutation algebra, it equals

$$-F^2(n,m)F^2(q,p)\langle 0| b_n \star b_m \star V \star a_q^\dagger \star a_p^\dagger |0\rangle$$

Once *red* $= 1$, Hallelujah!

$$red = -e^{-i\chi[(E_n J_m - E_m J_n) + (E_q J_p - E_p J_q)]}$$

V-independent

$\chi \to 0$, either $E_n, E_q, ...$ or $J_n, J_q ...$ should be extremely large.

How to account for it? Resort to the period of phase factor.

$$J \to \frac{J}{\hbar} \sim \text{regular number, say, n}$$
$$E \to E' + \frac{2\pi}{n\chi}$$

then

$$e^{-\frac{i}{2}\chi JE} \to e^{-\frac{i}{2}\chi n(E' + \frac{2\pi}{n\chi})} \to -e^{-\frac{i}{2}\chi JE'}$$

Keep p,q, n regular, set E_m large enough, and E'_m is the observed energy.

A fermion(boson) with large enough energy or/and angular momentum can effectively turn the system into the bosonic(fermionic) with regular energy or/and angular momentum.

Tunnelling?

Somehow, $\frac{2\pi}{\chi}$ *is like an energy barrier, and the non-zero overlap could be viewed as the tunneling effect.*

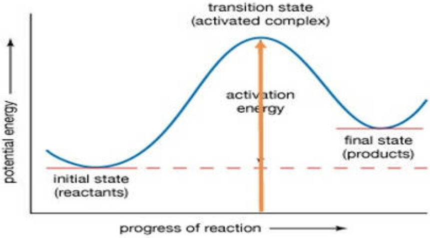

Figure 1: barrier

The 3-particle final state

The final bosonic state is:

$$
\begin{aligned}
|\phi\rangle =& \frac{1}{\sqrt{3!}}\{2|1+,1-,1+\rangle + 2e^{i\chi E_{1s}}|1+,1+,1-\rangle \\
& +2e^{-i\chi E_{1s}}|1-,1+,1+\rangle\}
\end{aligned}
$$

Transition Amplitde

$$A_\chi = \langle \phi | V | \psi \rangle$$

Final state: $1S \Rightarrow$ The dominant channel: $2P \to 1S$.

The transition in regular spacetime is prohibited not because of the disability of the potential, but rather of the vanishing final state due to spin-statistics.
We can still use the regular V.

V: the SM potential inducing the particle to jump from N orbit to N-i orbit, and
only relative to particles in$N-1, N-2, N-3...orbits$.

The only non-trivial transition: $X(2P) \to 1S$
A typical term would be like:

$$
\begin{aligned}
& {}_{\hat{m}}\langle 1+, 1-, 1+|V|1+, 1-, X+\rangle_{\hat{n}} \\
\equiv & {}_{\hat{m}}\langle 1+|1+\rangle_{\hat{n}}\, {}_{\hat{m}}\langle 1-|1-\rangle_{\hat{n}}\, {}_{\hat{m}}\langle 1+|V|X+\rangle_{\hat{n}} \\
= & {}_{\hat{m}}\langle 1+|1+\rangle_{\hat{n}}\, {}_{\hat{m}}\langle 1-|1-\rangle_{\hat{n}}\, {}_{\hat{m}}\langle +|+\rangle_{\hat{n}} \langle 1|V|X\rangle
\end{aligned}
$$

Note that

$$|X\rangle = \frac{1}{2}(|X+\rangle_{\hat{n}} + |X-\rangle_{\hat{n}})$$

Recover another quantum number I:

$$|X+\rangle_{\hat{n}} \to |X+, I = -1, 0, 1\rangle_{\hat{n}}$$

average $J_X = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$$\begin{aligned} |X+\rangle_{\hat{n}} :&= \frac{1}{4}(|X,+,I_n = -1\rangle_{\hat{n}} + 2|X,+,I_n = 0\rangle_{\hat{n}} + |X,+,I_n = 1\rangle_{\hat{n}}) \\ &= |+\rangle_{\hat{n}}\frac{1}{4}(|X, I_n = -1\rangle + 2|X, I_n = 0\rangle + |X, I_n = 1\rangle) \end{aligned}$$

Therefore,

$$|X\rangle \equiv \frac{1}{8}\sum_{all}|X, s=\pm\frac{1}{2}, l=-1,0,1\rangle_{\hat{n}}$$

One then could read $\langle 1|V|X\rangle$ as

$$\sum_{J,J'}\frac{1}{8}\langle 1, J|V|X, J'\rangle$$

the summation of the final angular momentum configurations and the average of the initial angular momentum configurations, which is the regular transition amplitude detected by experiments.

Apply the spin-overlap:

$$_m\langle\alpha|\beta\rangle_n = \frac{1}{2}[1 + (-1)^{\alpha-\beta}\hat{m}\cdot\hat{n}]$$

Integrate the direction:

$$\int \frac{d\Omega_m}{4\pi} m_i = 0, \int \frac{d\Omega_m}{4\pi} m_i m_j = \frac{1}{3}\delta_{ij}$$

$$
\begin{aligned}
&3!\frac{-iA_\chi}{\langle 1|V|X\rangle} \\
= &\frac{1}{6}[\sin(\chi E_{1s}) + \sin(2\chi E_{1s}) + \sin(3\chi E_{1s})] \\
&-\frac{1}{3}[\sin(\chi\frac{E_{1s}+E_X}{2}) + \sin(\chi\frac{3E_{1s}+E_X}{2})] \\
&+\sin[\chi(E_{1s}-E_X)] \\
&\sim \frac{4}{3}\chi\Delta E
\end{aligned}
$$

Discussion

Surely, in regular spacetime one can also construct a bosonic state for electrons, which is little more than turning "-" into "+". However, such states are **neither accountable nor approachable.**

In the non-commutative spacetime, the non-zero overlap undoubtedly implies its existence since
everything that could happen quantum mechanically happens.

Better, but not enough. Future works await.

Introduction Redux

- What's the meaning of 'exotic'?

  It violates PEP partly that although identical fermions are still forbidden, fermions can share the same state via turning into the same-type bosons.

- Why could non-commutativity induce exotic transitions?

  The complex phase structure induced by non-commutativity causes the non-zero overlap. The potential is not necessarily beyond the SM, it's the state that opens the new vista.

END.

Thanks!