T-duality

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1 T-Duality

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1.1 Preamble

"T" means "Toroidal," and literally, this duality is the duality of a circle.

Before we get into the T-dual, let's clarify a few subtle points.

1: One should not understand the intrinsic length of a string as naively as the length used in everyday life. It is rather closer to a topological property - a mere parameter. One can imagine strings living in the world of Platonic ideas, and its projection onto the real world is what we generally mean by length, which is responsible for the string tension. A string can be shorter than an atom or longer than a galaxy, provided the energy is enough.

2: The core of T-duality is a curled-up dimension, which was first proposed by Kaluza and Klein in the 1920s to unify the electromagnetic and gravitational fields. Einstein liked this idea but later found the radius of this dimension should be fixed, which however violates the essence of GR: the dynamic spacetime. Even accepting that this dimension is uniquely determined, it is still unstable. Einstein eventually gave up on this idea. It became a beautiful mistake in history.

3: We will mainly talk about curled one-dimensional space, so to avoid tedious notation, we always omit the indices for regular dimensions.

4: Although it is intuitive to say that one dimension is curled into a circle, sometimes it can cause some confusion because it is too heuristic. When thinking of a circle, we usually embed it into a higher-dimensional space, and wrongly imagine a projection onto it. We shall thus use the term "periodic dimension" or "compactification" to describe them.

1.2 The Amazing Adventure of Closed Strings

We generally do not curl the time dimension to avoid causal issues. Take $2\pi R$ as the period of space dimensions, and then we discuss the closed string as an example.

1.2.1 Winding Number

Stretching a string, wrapping it m times around the compactified dimension, and then splicing the beginning and end into a piece, we can produce a winding closed string, which cannot contract continuously to a point. m is a topological invariant. The real length of the string is $2\pi Rm$ with the intrinsic energy being $w = T \times 2\pi Rm = \frac{mR}{\alpha'}$.

It also has momentum or angular momentum since it is now a ring. It is determined by the Bohr quantization condition $L = n\hbar$, $n \in N^+$, so $p = \frac{L}{R} = \frac{n}{R}$, where the natural unit is used, which can also be obtained by using the group transformation by calibrating a state $|x\rangle$ with position eigenvalues and translating $2\pi R$ along the x-direction as $e^{ip2\pi R}|x\rangle = |x + 2\pi R\rangle = |x\rangle$, so $p2\pi R = 2\pi n$ and $p = \frac{n}{R}$.

So far, in addition to the vibrational term, we have two quantized energies, $p = \frac{n}{R}$ and $w = \frac{mR}{\alpha'}$. The former also occurs in point-like particles, but the latter, winding energy, is unique and yields interesting properties.

Surely, one might wonder what if we rolled up two or more dimensions simultaneously? Would there be somewhat extended T-duality? The answer depends on whether the manifold has a "hole" or not. If two dimensions form a smooth sphere, a one-dimensional closed string can always contract continuously to a point, just like in flat spacetime. When there is a "hole," let's say the simplest "donut," it does not necessarily contract to a point.

Does the number of holes affect the T-duality?

1.2.2 equation of motion

The other dimensions are normal and disjoint, of which the solutions are thus the same as the normal closed string solutions.

For the compactified dimension, there is a special periodic boundary condition: $X|_{\sigma=0} = X|_{\sigma=2\pi} + 2\pi mR$, compared with what we have solved before, $X|_{\sigma=0} = X|_{\sigma=2\pi}$, one can see a mere additional constant.

Now, as usual decomposing X into left and right waves: $X = X_L(\tau + \sigma) + X_R(\tau - \sigma) = X_L(\sigma^+) + X_R(\sigma^-)$, the period condition is: $X_L(\sigma^+ +$

$2\pi) - X_L(\sigma^+) = X_R(\sigma^-) - X_R(\sigma^- - 2\pi) + 2\pi mR$. The constant term is irrelevant since each side would cancel out mutually. The vibrational term is related to the second derivative, so it should also be the same. Therefore, only can the first-order term reflect the difference:

$$X_L(\sigma^+) = \frac{1}{2}x_0^L + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n}{n}e^{-in\sigma^+}$$
$$X_R(\sigma^-) = \frac{1}{2}x_0^R + \sqrt{\frac{\alpha'}{2}}\alpha_0\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-in\sigma^-}$$

From the period condition, we have $\tilde{\alpha}_0 - \alpha_0 = \sqrt{2\alpha'}w$, this is the difference between the first-order term, which should be zero for the regular case as required by the level matching: the constant relates to the center of mass coordinates, and the center of mass of left and right waves are naturally the same for regular closed strings.

The winding, however, will affect it, so x_0^L and x_0^R are generally not equal. Nontheless, given that the total constant term of $X_L + X_R = X$ should be the center of mass coordinate x_0, we have: $x_0^L = x_0 + q_0$, $x_0^R = x_0 - q_0$. Therefore,

$$\begin{aligned} X &= X_L + X_R \\ &= x_0 + \sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}_0 + \alpha_0)\tau + \sqrt{\frac{\alpha'}{2}}(\tilde{\alpha}_0 - \alpha_0)\sigma + ... \end{aligned}$$

This result may look puzzling, but let's calculate the mass-center momentum: $p = T\int \dot{X}d\sigma = T\int(\dot{X}_L + \dot{X}_R)d\sigma = \frac{1}{\sqrt{2\alpha'}}(\tilde{\alpha}_0 + \ alpha_0)$, and by comparing it with another thing with the same unit: $w = T\int(\dot{X}_L - \dot{X}_R)d\sigma = \frac{1}{\sqrt{2\alpha'}}(\tilde{\alpha}_0 - \alpha_0)$, we have: $X = x_0 + \alpha' p\tau + \alpha' w\sigma + ...$

An interesting term appear - two quantized momenta, one related to τ and another to σ. As we will see later, the T-duality in some measure means the equivalence between these two momenta.

1.2.3 M^2-spectrum

In string theory, the observable measure depends on the spectrum of the M^2 operator, so let's discuss M^2.

The string constraint, $T_{\alpha\beta} = 0$, still has to be satisfied and has nothing to do with the boundary conditions, so we have

$$\begin{aligned}
(\partial_+ X_L)^2 &= 0 \\
&= (\sqrt{\frac{\alpha'}{2}} \sum_n \tilde{\alpha}_n e^{-in\sigma^+})(\sqrt{\frac{\alpha'}{2}} \sum_p \tilde{\alpha}_m e^{-im\sigma^+}) \\
&= \frac{\alpha'}{2} \sum_p \sum_n \tilde{\alpha}_{p-n} \tilde{\alpha}_n e^{-ip\sigma^+} \\
\Rightarrow L_n &= \frac{1}{2} \sum_n \tilde{\alpha}_{p-n} \tilde{\alpha}_n = 0 \\
\Rightarrow L_0 &= \frac{1}{2} \tilde{\alpha}_0^2 + \frac{1}{2} \sum_{n<0} \tilde{\alpha}_{-n} \tilde{\alpha}_n + \frac{1}{2} \sum_{n>0} \tilde{\alpha}_{-n} \tilde{\alpha}_n \\
&= \frac{1}{2} \tilde{\alpha}_0^2 + (\tilde{N} - 1) = 0
\end{aligned}$$

where the conclusion of normal ordering is used. It is important to note that constrain is a squared term, which implies the summation of indices, involving all the flat and compactified dimensions.

For the compactified dimension, according to the equations of p and w, the inverse solution is: $\tilde{\alpha}_0 = \sqrt{\frac{\alpha'}{2}}(p + w)$, and the $\tilde{\alpha}_0$. One can see the dependence of the flat dimension remains unchanged, so the above equation is written again as: $\frac{\alpha'}{4}[\sum_{flat} p^2 + (p + w)^2] + (\tilde{N} - 1) = 0$, that is.

$$\begin{aligned}
M^2 &= (p + w)^2 + \frac{4}{\alpha'}(\tilde{N} - 1) \\
&= (\frac{n}{R})^2 + (\frac{mR}{\alpha'})^2 + \frac{2}{\alpha'}(2\tilde{N} + mn - 2)
\end{aligned}$$

The energy of particles living in a flat space should not be dependent on the compactified dimension, so $M^2 = -\sum_{flat} p^2$.

Similarly, we get the spectrum for $(\partial_- X_R)^2 = 0$: $M^2 = (\frac{n}{R})^2 + (\frac{mR}{\alpha'})^2 + \frac{2}{\alpha'}(2N - mn - 2)$, adding the two M^2: $M^2 = (\frac{n}{R})^2 + (\frac{mR}{\alpha'})^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)$. By comparison, $N + \tilde{N} = 2\tilde{N} + mn$, i.e. $N - \tilde{N} = mn$. It is the second difference, which means that the level of the left and right waves can differ by an integer multiple, which is related to the momentum and winding number.

big circle small circle indistinguishable According to the already obtained M^2 property, we find that doing.

$$R \Longleftrightarrow \frac{\alpha'}{R}$$
$$m \Longleftrightarrow n$$

M^2 is unchanged, it is easy to verify that at this time, $p' = \frac{mR}{\alpha'} = w, w' = \frac{n}{R} = p$. That is, exchanging p and w does not change the mass (or energy), but what does this mean? What does it mean?

Recall what we got before: $X = x_0 + \alpha' p\tau + \alpha' w\sigma + ...$ Denote the spacetime coordinates after exchanging p and w are $X' = x_0' + \alpha' w\tau + \alpha' p\sigma +$ ..., which implies that if we will obtain two identical coordinate systems, X and X', by only knowing the energy spectrum. That there are two equivalent interpretations of a physical phenomenon is called duality. The radiuses of the compactified dimensions are inverse, and then large and small circles are physically equivalent. The T-duality.

However, how to justify the x_0'? It is in principle different from x_0 because the dimensional sizes have changed, and the center-of-mass coordinates are naturally different. But what exactly is the difference?

Notice that the new first-order term can be constructed from the original X_L and X_R: we define $Y = X_L - X_R = q_0 + \alpha' w\tau + \alpha' p\sigma + ...$ Y and X' should be the same since they have the same spacetime dimensions and the same behavior of the closed string, so $x_0' = q_0$, that is, the center-of-mass coordinates also vary due to the T-duality, and we replace X' by Y afterward.

The classical analogy is regular in everyday life: a thick stick struck at a low speed is not different from a thin stick struck at a high speed when the pressure difference is not taken into account.

Note that the equivalence of these two in string theory comes from the equivalence of the mutualy inverse radius.

1.3 The Amazing Adventure of Open Strings

Previously we only discussed the relation between closed strings and compactified dimensions. Now let's consider open strings.

However, unlike closed strings, the special property of open strings comes from the fact that their endpoints have to be fixed on $D_p - brane$, so the T-duality for open strings is a relation between $D_p - brane$.

1.3.1 $D_p - brane$

Imagine an N+1-dimensional spacetime, still compactify some dimensions as a supercolumnar surface. Assuming that the whole supercolumnar surface is a D-brane, i.e., $D_N - brane$, the string can move freely in the space, except that the momentum in the compactified dimension is quantized: $p = \frac{n}{R}$, and the meaning of each parameter is the same as before.

Since the open string can move freely, it can always contract continuously to a point. There is no winding number, hence no corresponding energy w. So the spectrum is $M^2 = (\frac{n}{R})^2 + \frac{2}{\alpha'}(N-1)$.

But if we directly dualize the compactified dimension, i.e. $\frac{n}{R} \Rightarrow \frac{mR}{\alpha'}$, the spectrum is different from the original one. The T-duality of open strings needs correction - the D-branes. First, dualize the radius of the compactified dimension, and at the same time, fix the coordinates of the open strings in this new tight dimension (change the boundary condition to DD-type) so that it cannot move on this circle, at which point $D_N - brane$ becomes $D_{N-1} - brane$.

In this way, it has no momentum, p=0. But meanwhile, since its endpoint is fixed on the curl dimension, it may be useful to write down the endpoint coordinates $X = 0$, which is periodized, and then $X = 2\pi m R'$, which intuitively looks like a straight line divided into an infinite number of intervals of length $2\pi R$, and the endpoints of the string can be connected at the endpoints of any interval.

However, since the endpoints are restricted to these points, it is impossible to move freely to contract to a point. Here the winding numbers conjure up.

Now, let us prove that the true length of an open string placed in this way is $2\pi m R'$. Denote the left and right waves of the open strings solved by the periodized boundary condition as X_L and X_R. Suppose the total coordinates is $X = X_L + X_R$ in the original $D_N - brane$ spacetime.

Inspired by the closed strings, we define the coordinates of the compact dimension in the dual spacetime, i.e., $D_{N-1} - brane$, as Y=X$_L - X_R = q_0 + \sqrt{2\alpha'}\alpha_0\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in\tau} sinn\sigma$. Then.

$$Y(\tau,\pi) - Y(\tau,0) = \sqrt{2\alpha'}\alpha_0\pi = 2\pi mR'$$

where the original open string momentum $p = \frac{\alpha_0}{\sqrt{2\alpha'}} = \frac{m}{R}$ is used, as well as the radius $R' = \frac{\alpha'}{R}$ after doing the dual transformation, so there is the intrinsic energy $w = T \times 2\pi mR' = \frac{mR'}{\alpha'}$, so $M^2 = (\frac{mR'}{\alpha'})^2 + \frac{2}{\alpha'}(N-1)$, which is dual to the previous case of $D_N - brane$.

It is the case of the T-duality for the open string.

1.4 Summary

We have not considered the commutator, electromagnetic field, and whatnot. The very focus is the free bosonic string, which will not be affected by D-branes. Thus the T-duality of the closed strings is not destroyed.

In summary, when

$$R \Longleftrightarrow \frac{\alpha'}{R}$$
$$m \Longleftrightarrow n$$
$$D_p - brane \Longleftrightarrow D_{p-1} - brane$$

the physics is unchanged. It is the T-duality.

Note that even though it says $D_{p-1} - brane$, several dimensions being compactified simultaneously are allowed, which could be called the donut manifold.

Furthermore, T-duality tells us the open strings in a two-hole donut are equivalent to the one being fixed on a point. Strange, but it is true.

1.5 Hodge-Duality and T-Duality

Consider a spacetime with a compactified dimension of $R \to 0$ that can barely move in this direction, and which, according to the T-duality, is physically equivalent to $R \to \infty$, i.e., unfolding it infinitely so that the spacetime becomes an ordinary flat spacetime. The strings moving on a minimal ring are equivalent to that on an ordinary spacetime.

How do we describe this unfolded dimension? Recall the previous work that if X is the spacetime coordinate of that $R \to 0$ dimension and Y is the spacetime coordinate of the unfolded dimension, and: $\partial_\tau X = \partial_\sigma Y, \partial_{sigmaX=\partial_\tau Y}$.

Note $partial_\alpha X = F_\alpha$ is a two-dimensional vector with component 0 being p and component 1 being w, and $\partial_\alpha Y = \tilde{F}_\alpha$ is also a two-dimensional vector with component 0 being w and component 1 being p. The two vectors are equal in the sense of T-duality, which is another way of interchanging p and w writing.

Introducing the two-dimensional Levi-Civita tensor $\epsilon_{\alpha\beta}$, the above equation can be expressed as $F_\alpha = \epsilon_{\alpha\beta}\tilde{F}^\beta$, i.e., these two vectors are linked by the Hodge dual, which shows that they have orthogonality since it is easy to verify $F_\alpha \tilde{F}^\alpha \equiv 0$.

Also, since $\partial_\alpha X = dX$, then $ddX = dF_\alpha = 0 = \partial_\tau w - \partial_\sigma p, ddY = d\tilde{F}_\alpha = 0 = \partial_\tau p - \partial_\sigma w = \partial_\tau^2 X - \partial_\sigma^2 X$. The first term is constant and is determined by the structure of the theory itself, while the latter is the original equation of motion, or the conservation of energy when there is no source (the constrain of $T_{\alpha\beta} = 0$ is given by the equation of motion).

Thinking of the Maxwell equation in a vacuum, which is very similar in structure to the two conclusions above, and since the electromagnetic dual is a 4-dimensional tensor Hodge dual, there is an analogy: ($\tilde{F}$ rewritten as $*F$)

$$dF_\alpha = 0 \leftrightarrow dF_{\alpha\beta} = 0$$
$$d * F_\alpha = 0 \leftrightarrow d * F_{\alpha\beta} = 0$$

In this sense, p and w are much like E and B. The size dual is very similar to the electromagnetic dual. Did kaluza and klein have this in mind when they built their theory? Is the so-called T-duality a two-dimensional Hodge duality? Or for ten dimensions, is there such a Hodge dual?

Can this analogy be extrapolated? If we are talking about free bosonic strings at the moment, does the fact that there are ”no magnetic monopoles” when there are interactions affect the physics of string theory to two mutually dual spacetimes?

1.6 Rescale

The equivalence of this size circle is a rare thing. Only with sand, you can make a bunker sizing centimeters, but it is impossible to make a bunker sizing meters due to the not-invariance of gravity, pressure, etc. Galileo thought of this and believed that it is as important as his discovery of the "principle of equivalence of inertial systems."

So, in the everyday physical world, basically no such equivalence under different dimensions, otherwise there is no need for quantum mechanics or general relativity, and Newtonian mechanics can dominate the world. The equivalence of small and large circles is a unique property of strings.

1.7 Further Questions

- T-duality also restricts the limiting size of dimensions since $R \leftrightarrow \frac{\alpha'}{R}$ is physically invariant, then everything $R < \sqrt{\alpha'}$ is equivalent to $R > \sqrt{\alpha'}$, so $\sqrt{\alpha'}$ seems to be the smallest size in this sense. It happens to be the same magnitude as the Planck length.

- So far, the T-duality has involved the positive curvature as well as zero curvature, but does it hold for negative curvature as well? For example, the AdS spacetime. From the spectrum, the radii involved in M^2 are square terms, and thus it seems impossible to distinguish between positive and negative curvature. If so, it is not only impossible to distinguish between small and large circles, but also impossible to distinguish between saddle and sphere... What if there is a geometry that deals with $R^2 + \frac{1}{R^2}$ instead of R? Would it be a the habitat of T-duality?

- From GR's point of view, spacetime is dynamic and can be curled given enough energy. Then the equivalence of radius implies the equivalence of the energy required to bend, which contradicts GR where different energy yields different curvature. Also, it seems that the graviton in string theory is a free propagating closed string, which is not affected by T-duality. how can T-parity be coordinated with GR?

- Let's assume that the original compactified radius changes ΔR, accordingly, $\Delta p = -\frac{n}{R^2}\Delta R$ and $\Delta w = \frac{m}{\alpha'}\Delta R$, adding these them together: $\Delta p + \Delta w = (\frac{m}{\alpha'} - \frac{n}{R^2})\Delta R$, assuming that $\Delta R > 0$, the total rate of change of these two energies is monotonically increasing, taking the zero point at $\sqrt{\frac{n\alpha'}{m}}$, converging to $\frac{m}{\alpha'}$, and after doing the dualization , $\Delta p + \Delta w = (\frac{n}{\alpha'} - \frac{mR^2}{\alpha'^2})(-\frac{\alpha'}{R^2}\Delta R) = (\frac{m}{\alpha'} - \frac{n}{R^2})\ DeltaR$, the total rate of change function is the same.

  However, in terms of the positive and negative of the function, the total energy always decreases first, has a minimum at $R = \sqrt{\frac{n\alpha'}{m}}$, i.e., when the two energies are equal, and then increases afterward.

  What should be the explanation for it?

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