GUP

Horan Tsui

May 2022

1 Abstract

When considering the action of gravity, the momentum of an object in a finite region cannot be infinite but will have an upper limit proportional to the size of the region. Based on this, Adler et al. have corrected the HUP, obtaining many interesting results by approximate methods. However, such corrections generally break the symmetry of position and momentum. In this paper, I try to give an extension that maintains the symmetry and can yield those interesting results in an analytical way. The extension also shows that the black hole undergoes a first-order phase transition when it reaches the Planck scale by evaporation, and then the temperature is linearly proportional to its mass and has negative entropy.

2 Introduction

The original argument of Adler et al is that considering a photon with an effective mass $\frac{p}{c}$ and an electron with mass m, then let them collide due to (Newtonian)gravity at distance L, the momentum theorem tells us: $\int G\frac{(p/c)m}{r^2}dt = m\Delta v$.

As an approximation, we use the force at the beginning to replace the real force which will become larger and larger as they are closer and closer, and set $\frac{L}{c} \approx \Delta t$, then: $G\frac{(p/c)m}{L^2}\frac{L}{c} \leqslant m\frac{\Delta X}{L/c}$. Since the momentum of photon is exactly the momentum variance of the electron, we have:

$$G\frac{\Delta p}{c^3} = \frac{l_p^2}{\hbar}\Delta p \leqslant \Delta X \tag{1}$$

where $l_p = \sqrt{\frac{\hbar G}{c^3}}$ is Planck length.

The formula (1) is the outcome considering the effect of gravitational collision.

Another way to think it is that considering a box with limited size, Δx. Quantum mechanics tells us the minimal momentum this box could have is $\frac{\hbar}{\Delta x}$, that is, a quantum particle with the de Broglie wavelength Δx, while GR tells us the maximal momentum is carried by the black hole whose horizon size is Δx. Suppose it's a Schwarzschild black hole, then immediately, we have $\Delta x = \frac{Gm}{c^2} = \frac{G(p/c)}{c^2}$, as same as formula (1).

The traditional way of revamping HUP is adding (1) into it literally, $\Delta x \geqslant \frac{\hbar}{\Delta p} + \frac{l_p^2}{\hbar}\Delta p$. By further considering string theory, AdS spacetime and whatnot, they have added more terms of higher order of Δx and Δp, for instance: $\Delta x\Delta p \geqslant \hbar(1+\alpha\Delta p+\alpha^2\Delta p^2+\beta\Delta x+\beta^2\Delta x^2)$, which is called LEGUP. Nevertheless, I thought these extra terms are physically obscure.

Other than this, there are two unsatisfactory points. Firstly, this extension breaks the readily accepted symmetry between momentum and position. Secondly, the order of $\hbar \approx 10^{-34}$, while $\frac{G}{c^2}$ is 10^{-35}. The equality of two factors turns a reciprocal function($\Delta p > \frac{\hbar}{\Delta x}$) into a quadratic function($\frac{l_p^2}{\hbar}\Delta p^2 + \hbar - \Delta p \Delta x > 0$), the effect must be unneglectable, it does not only rectify HUP, but the whole structure of the quantum world, and I'm personally willing to keep HUP. It might also be viewed as a perturbation but the appearance of an extra small pre-factor is required, which would propose another question.

Based on these, I thought we should view the effect of gravity from another aspect.

3 Extension

We insist three tenets:

- keep HUP, $\Delta x \geqslant \frac{\hbar}{\Delta p}$
- add (1), $\Delta x \geqslant \frac{l_p^2}{\hbar}\Delta p$
- keep the balance of x and p

then I found a quite simple formula satisfying these tenets(after dimensionless-lize):

$$\alpha \frac{l_p}{\Delta x} + \frac{\Delta x}{l_p} \geqslant \frac{\hbar}{l_p \Delta p} + \beta \frac{l_p \Delta p}{\hbar} \tag{2}$$

If we treat is as a perturbation, then (2) becomes

$$\Delta x \Delta p \geqslant \hbar + \beta \frac{l_p^2}{\hbar}\Delta p - \alpha l_p^2 \frac{\Delta p}{\Delta x}$$

where the minus sign could be absorbed into α. This amounts to offer a term like $\frac{\Delta p}{\Delta x}$, which is ugly and meaningless. We need to view it from another aspect.

Redefine: $\frac{\Delta x}{l_p} = a, \frac{l_p}{\hbar}\Delta p = b$, then (2) becomes:

$$\frac{\alpha}{a} + a \geqslant b + \frac{\beta}{b} \tag{3}$$

multiply "ab" at both side, $ab(a-b) \geqslant \beta a - \alpha b$, when

- $\alpha = \beta > 0$
- $a - b \geqslant 0$

are satisfied, (3) becomes $ab \geqslant \beta$, which is exactly $\Delta x \Delta p \geqslant \beta\hbar$.

The second requirement, $a - b \geqslant 0$, is actually $\Delta x \geqslant \frac{l_p^2}{\hbar}\Delta p$, the bar imposed by gravity.

So in my viewpoint, equation (2), or (3), is not really a correction of the HUP but an extension when gravity is taken into account. Such a strange form will expand the space of solutions and the bar of gravity will cancel parts of solutions, and the remaining new solution is what we are interested in.

The value of β is not critical because, by the experience from quantum mechanics, different systems will give different factors, for example, the uncertainty relation of the S.H.O. is $\Delta x \Delta p \geqslant \hbar(n + \frac{1}{2})$. For simplicity, we set it to 1.

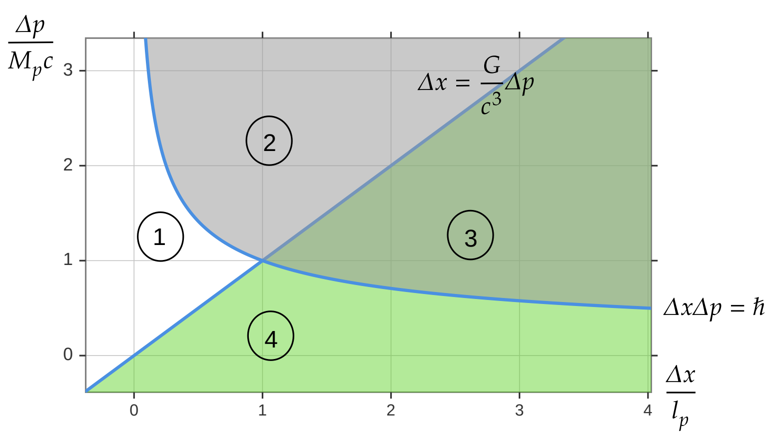
4 Analytic

When we focus on "a", (3) would write as: $a^2 + 1 - (\frac{1}{b} + b)a \geqslant 0$, and the solution is $a \leqslant min[b, \frac{1}{b}] \cup a \geqslant max[b, \frac{1}{b}]$;

When we focus on "b", (3) is: $b^2 + 1 - (a + \frac{1}{a})b \leqslant 0$, and the solution is $min[a, \frac{1}{a}] \leqslant b \leqslant max[a, \frac{1}{a}]$.

For example, let $\Delta x = \frac{Gm}{c^2}$, when $m > M_p$ (M_p is the Planck mass), then $\frac{\hbar}{Gm/c^2} \leqslant \Delta p \leqslant mc$, when $m < M_p$, $mc \leqslant \Delta p \leqslant \frac{\hbar}{Gm/c^2}$.

The final solution is plotted as follows:



Remarks about this figure:

- The blue diagonal line is $\Delta x = \frac{G}{c^3}\Delta p = \frac{l_p^2}{\hbar}\Delta p$, i.e. the upper limit of the bar exerted by gravity, and the regions satisfying this constraint are the green regions on the right (regions 4 and 3).
- The blue curve is $\Delta x \Delta p = \hbar$, i.e., the classical HUP limit, and the shaded regions above it,

2 and 3, are the regions of solutions allowed by the original HUP.

- The solution of (3) are the regions 1 and 3, a dual diagram that $\forall \Delta x_1 < l_p, \exists \Delta x_2 > l_p$, satisfying $\Delta p_1 = \Delta p_2, \Delta x_1 \Delta x_2 = l_p^2$.

  It's easy to check when $\Delta x = \frac{Gm}{c^2} > l_p$, the upper and lower limit of Δp are exactly what we said before.

- Considering the restriction by the bar of gravity, the final practical solution is: the blue straight line at the junction part of region 1 and region 4, plus region 3 and its boundary.

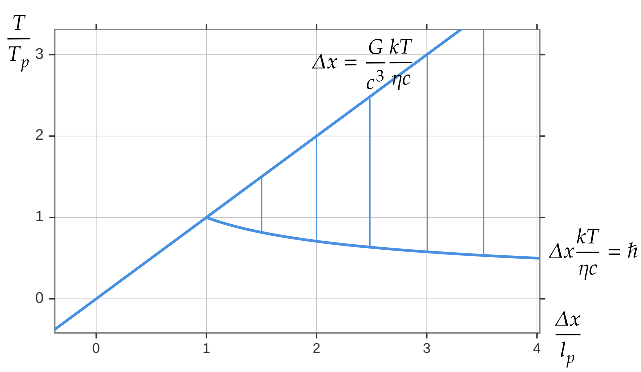
We see that the most important thing about the whole extension is that it gives a completely new solution when $\Delta x < l_p$, while the original HUP solution in this zone is the gray region above, namely, Δp becomes extremely large.

The gravity plays an indispensable role in this small scale, no longer making Δp explode due to quantum fluctuations, but they two work together to restrict Δp to be linear with Δx.

5 Thermodynamics

Given the formula, $kT = \eta c \Delta p$, where when $\eta = \frac{1}{4\pi}$ this formula gives exactly the Hawking temperature and we treat it as a factor now, we can go into the thermodynamics.

Firstly, there is a $T - \Delta x$ diagram:



where T_p is the Planck temperature and the blue line boundary and the blue vertical line area are the solution space.

We now illustrate it with the black hole, namely assuming $\Delta x \equiv \frac{Gm}{c^2}$.

- When $\Delta x > l_p$, the lower limit of temperature is $kT = \eta \frac{\hbar c^3}{Gm}$, which is exactly the Hawking temperature, which means the black hole is the coldest object of the same size. Also, there is an upper limit to the temperature due to gravity, which is $kT = \eta mc^2$, i.e. all the energy of the object becomes heat, which is intuitively acceptable.

- When $\Delta x = l_p$, the upper and lower temperature limits converge, reaching the Planck temperature T_p.
- When $\Delta x < l_p$, the temperature takes only one form, $\eta mc^2/K$, which means that the temperature decreases linearly with size when the black hole evaporates to a size smaller than the Planck scale. This similar result was previously obtained by Carr et al. in 2016, but mainly by approximation.

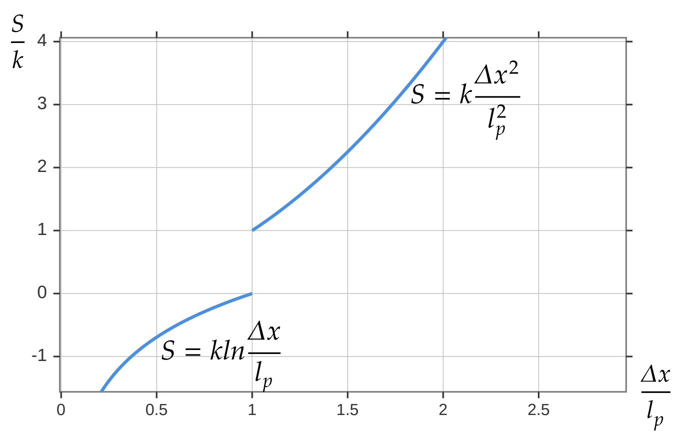
5.1 Entropy

In the following part, we will choose to use either Δx or $m = \frac{c^2 \Delta x}{G}$ as the variable according to convenience.

when $\Delta x < l_p$, using $\frac{\partial mc^2}{\partial S} = T$, which gives:

$$S = kln\frac{m}{M_p} = kln\frac{Gm/c^2}{l_p} = \frac{\Delta x}{l_p} \tag{4}$$

When $\Delta x > l_p$, the entropy of the black hole is just Bekenstein-Hawking entropy, $S = k(\frac{m}{M_p})^2 = k\frac{A}{4l_p^2} = k\frac{(Gm/c^2)^2}{l_p^2} = \frac{\Delta x^2}{l_p^2}$, so we have the $S - \Delta x$ diagram:



When $\Delta x < l_p$, S becomes negative. I'm not sure how to explain that exactly. (in quantum entangled states, the conditional entropy is indeed negative?)

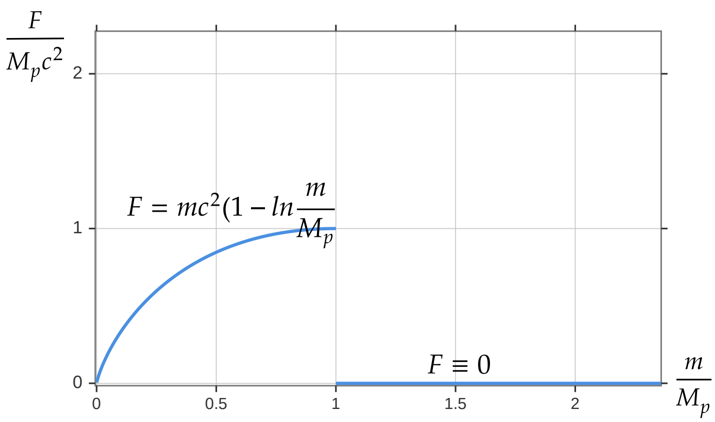
There is a subtle point that the RT formula is calculated under the AdS/CFT dual, however the black hole does not evaporate in AdS spacetime, keeping its scale as same as the initial. It consequently cannot be smaller than l_p, because for those black holes created by quantum fluctuations, Δp is inversely proportional to Δx, and when Δx reaches l_p it will become a black hole, and then continue to increase the energy will increase its size, for those coming from the celestial evolution, the size are evidently larger than l_p. It seems that the only way to probe the sub-planckian region is by the evaporation of black holes

5.2 Phase transition

At $\Delta x = l_p$, there is an entropy jump implying the latent heat: $Q = T\Delta S = T_p k = M_p c^2$.

Consider the heat capacity, when $\Delta x > l_p$, $C = \frac{U}{T} = \frac{\partial mc^2}{\partial \eta \frac{\hbar c^3}{Gmk}} = -\frac{Gk}{\hbar c \eta}\frac{1}{m^2}$, when $\Delta x < l_p$, $C = \frac{mc^2}{\eta mc^2/k} = k$, so at l_p, the heat capacity jumps from a negative value to a positive constant k, hence indeed a phase transition.

We can also calculate the free energy. When $\Delta x > l_p$, $F = U - TS = mc^2 - \eta \frac{\hbar c^3}{Gm}\frac{1}{k}\frac{k}{\eta}(\frac{m}{M_p})^2 \equiv 0$, and when $\Delta x < l$, $F = mc^2 - mc^2 ln \frac{m}{M_p} > 0$. Here is the $F - m$ diagram:



So on reaching the Planck mass, the black hole does not spontaneously continue to shrink and needs to put into additional work $M_p c^2$, which reduces the entropy of the black hole and then releases $M_p c^2$ of heat.